In order to show that $m \geq 2$, for arbitrary $\epsilon > 0$ we present equilateral triangles $T_1, \ldots, T_5$ with side lengths $a_1 \leq \cdots \leq a_5$ such that $a_1^2 + \cdots + a_5^2 > 2 - \epsilon$ that cannot cover T. Let

$$\delta = \frac{\epsilon}{20}\,, \qquad a_1 = a_2 = a_3 = \delta\,, \qquad a_4 = a_5 = 1 - 5\delta\,.$$

Indeed,

$$a_1^2 + \cdots + a_5^2 > 2 - 20\delta = 2 - \epsilon\,.$$

Let us pin congruent equilateral triangles U, V, W with side length 2δ to the vertices of T as shown in figure 2. On applying T_3 and T_4 to T, each of T_3 and T_4 can meet at most one of the triangles U, V, W. Hence, there is a triangle among U, V, W which is not met by T_3 and T_4, say U has this property. But the combined area of T_1, T_2, T_3 is less than the area of U, so T_1, T_2, T_3 cannot cover U. This completes both our counterexample and the proof that $m \geq 2$.

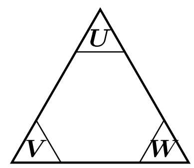

Figure 2

Next we look at solutions for the 48$^{\text{th}}$ IMO Bulgarian Team, First Selection Test, given at [2010: 275].

1. The sequence $\{a_i\}_{i=1}^{\infty}$ is such that $a_1 > 0$ and $a_{n+1} = \frac{a_n}{1+a_n^2}$ for $n \geq 1$.

(a) Prove that $a_n \leq \frac{1}{\sqrt{2n}}$ for $n \geq 2$;

(b) Prove that there exists n such that $a_n > \frac{7}{10\sqrt{n}}$.

Solved by Arkady Alt, San Jose, CA, USA; and Chip Curtis, Missouri Southern State University, Joplin, MO, USA. We give the solution by Alt.

Since $a_n > 0, n \geq 1$ then

$$a_{n+1} = \frac{a_n}{1+a_n^2} \iff \frac{1}{a_{n+1}^2} = \left(\frac{1}{a_n} + a_n\right)^2$$
$$\iff \frac{1}{a_{n+1}^2} = \frac{1}{a_n^2} + a_n^2 + 2 \iff b_{n+1} = b_n + 2 + \frac{1}{b_n}\,,$$

where $b_n := \frac{1}{a_n^2}$, $n \geq 1$ and we will prove:

a) $b_n \geq 2n$ for $n > 2$;

b) There is n such that $b_n < \frac{100n}{49}$.

a) Since $b_{n+1} = b_n + 2 + \frac{1}{b_n} > b_n + 2$ and $b_2 \geq 4$, by induction $b_n \geq 2n$.

b) Since $b_n \geq 2n$ for $n \geq 2$ then $b_{n+1} = b_n + 2 + \dfrac{1}{b_n} \leq b_n + 2 + \dfrac{1}{2n}$, $n \geq 2$ and, therefore,

$$b_{n+1} - b_2 = \sum_{k=2}^{n} (b_{k+1} - b_k) \leq \sum_{k=2}^{n} \left(2 + \frac{1}{2n}\right) = 2(n-1) + \frac{1}{2}(h_n - 1) ,$$

where $h_n = \sum\limits_{k=1}^{n} \dfrac{1}{n}$. Thus,

$$b_{n+1} \leq 2n - \frac{5}{2} + \frac{1}{2}h_n + b_2 < 2(n+1) + \frac{1}{2}h_{n+1} + b_2,\ n \geq 2$$
$$\Longrightarrow b_n < 2n + \frac{1}{2}h_n + b_2,\ n \geq 3 .$$

Note that $h_n < \sqrt{2n}$, $n \in \mathbb{N}$. Indeed, by the Cauchy Inequality we have

$$h_n^2 \leq n \cdot \sum_{k=1}^{n} \frac{1}{k^2}$$

and

$$\sum_{k=1}^{n} \frac{1}{k^2} < 1 + \sum_{k=2}^{n} \frac{1}{(k-1)k} = 1 + \sum_{k=2}^{n} \left(\frac{1}{k-1} - \frac{1}{k}\right) = 1 + 1 - \frac{1}{n} < 2 .$$

Since $\frac{100n}{49} = 2n + \frac{2n}{49}$ and $h_n < \sqrt{2n}$ then it suffices to prove that there is n such that $\frac{1}{2}\sqrt{2n} + b_2 < \frac{2n}{49} \iff 49b_2 < \sqrt{2n}\left(\sqrt{2n} - \frac{49}{2}\right)$.

It is easy to see that the latter inequality holds for any

$$n \geq n_0 = \max\left\{\frac{49^2 b_2^2}{2}, \frac{51^2}{8}\right\} .$$

Another variant of ending solution (b):

Since $2n < b_n < 2n + \dfrac{1}{2}\sqrt{2n} + b_2$ and $\lim\limits_{n\to\infty} \dfrac{2n + \frac{1}{2}\sqrt{2n} + b_2}{n} = 2$ then $\lim\limits_{n\to\infty} \dfrac{b_n}{n} = 2$ and, therefore, for any $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $\dfrac{b_n}{n} < 2 + \varepsilon \iff b_n < (2+\varepsilon)\, n$ for all $n > n_0(\varepsilon)$. In particular for $\varepsilon = \dfrac{2}{49}$ we have $b_n < \left(2 + \dfrac{2}{49}\right) n = \dfrac{100n}{49}$ for all $n > n_0\left(\dfrac{2}{49}\right)$.

That completes the *Corner* for this issue.